Differential formulation of Maxwell's equations

Laplace and Poisson eq. in electrostatics

The integral form of Maxwell's equations:

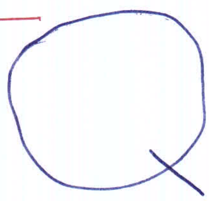
$$\begin{cases} \oiint \vec{E}\cdot d\vec{A} = \dfrac{Q_{int}}{\varepsilon_0} & \text{Gauss} \\ \oiint \vec{B}\cdot d\vec{A} = 0 & \\ \oint \vec{B}\cdot d\vec{l} = \mu_0\left(I + \varepsilon_0 \dfrac{\partial \Phi_E}{\partial t}\right) & \text{Ampère} \\ \oint \vec{E}\cdot d\vec{l} = -\dfrac{\partial \Phi_B}{\partial t} & \text{Faraday} \end{cases}$$

We use two mathematical formulas:

(1) $$\boxed{\oiint_{\Sigma} \vec{E}\cdot d\vec{A} = \int_V \nabla\cdot\vec{E}\, dV}$$

V is the volume.

Gauss-Ostrogradski



Σ surface including the volume V

transforms a surface integral in volume integral

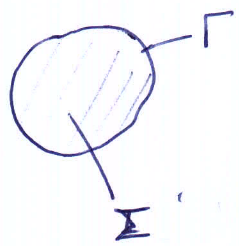
$$\nabla\cdot\vec{E} = \operatorname{div}\vec{E} = \frac{\partial E(x,y,z)}{\partial x} + \frac{\partial E(x,y,z)}{\partial y} + \frac{\partial E(x,y,z)}{\partial z}$$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$ nabla operator.

$\operatorname{div}\vec{E}$ = scalar product between ∇ and $\vec{E}$

② Stokes

$$\boxed{\oint_{\Gamma=\partial\Sigma} \vec{E}\cdot d\vec{l} = \iint_{\Sigma} (\nabla\times\vec{E})\cdot d\vec{A}}$$



$\Gamma = \partial\Sigma$ = contour of the surface Σ

a) We apply Gauss–Ostrogradski to 1st eq of Maxwell

$$\oiint \vec{E}\cdot d\vec{A} = \int_V \nabla\cdot\vec{E}\, dV = \frac{Q_{int}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_V \rho\, dV$$

$$\rho = \frac{dQ}{dV} \quad \text{= charge density}$$

$$\Rightarrow \boxed{\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$$

Gauss law in the differencial formulation for the electric field

Because: $\vec{E} = -\nabla V$

(electric field is the gradient of the electrostatic potential energy (=) electrostatic potential

$$\Rightarrow \nabla\cdot(\nabla V) = -\frac{\rho}{\varepsilon_0} \quad (\Leftrightarrow)$$

$$\boxed{\Delta V = \nabla^2 V = -\frac{\rho}{\varepsilon_0}}$$

Poisson equation

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{= Laplace operator}$$

2nd order differential equation

Knowing $\rho(x,y,z)$, by solving (e.g. numerically) the Poisson equation one can get the electric potential $V(x,y,z)$

no charge density source

If $\rho(\vec{r}) = 0$

$$\Rightarrow \boxed{\Delta V = 0}$$ Laplace equation

b) Applying Stokes to the 2nd eq of Maxwell:

$$\Rightarrow \oiint \vec{B} \cdot d\vec{A} = 0 \Rightarrow \boxed{\nabla \cdot \vec{B} = 0}$$

Gauss law for the magnetic field.

c) Applying Stokes for the Ampère's law $\Rightarrow$

$$\oint_{\Gamma} \vec{B} \cdot d\vec{l} = \oiint_{\Sigma} (\nabla \times \vec{B}) d\vec{A}$$

$$\nabla \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{B} = B_x \vec{i} + B_y \vec{j} + B_z \vec{k}$$

and $\quad I = \oiint \vec{J} \cdot d\vec{A}$ $\qquad$ $\vec{J}$ = current density

$$\frac{\partial \Phi_E}{\partial t} = \frac{\partial}{\partial t} \oiint \vec{E} \cdot d\vec{A}$$

$$\oiint (\nabla \times \vec{B})\, d\vec{A} = \oiint \left[\mu_0 \left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)\right] d\vec{A}$$

$$\Rightarrow \boxed{\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

differential Ampère

$$\frac{1}{c^2} = \mu_0 \varepsilon_0$$

$$\Rightarrow \boxed{\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}}$$

d) Applying Stokes for Faraday's law

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial \vec{B}}{\partial t} \Leftrightarrow \oiint_S (\nabla \times \vec{E})\, dA = \oiint \left(-\frac{\partial \vec{B}}{\partial t}\right) d\vec{A}$$

$$\Rightarrow \boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

differential Faraday

$$\Rightarrow \begin{cases} \nabla \cdot \vec{E} = \rho / \varepsilon_0 \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \end{cases}$$